Chapter 2
Modeling in the Frequency Domain

Goals

- Review the Laplace transform.
- Learn how to find the transfer function for electrical and mechanical system.
- Learn how to linearization a nonlinear system.

2.1 Introduction

This chapter represents an introduction to the method of modeling.

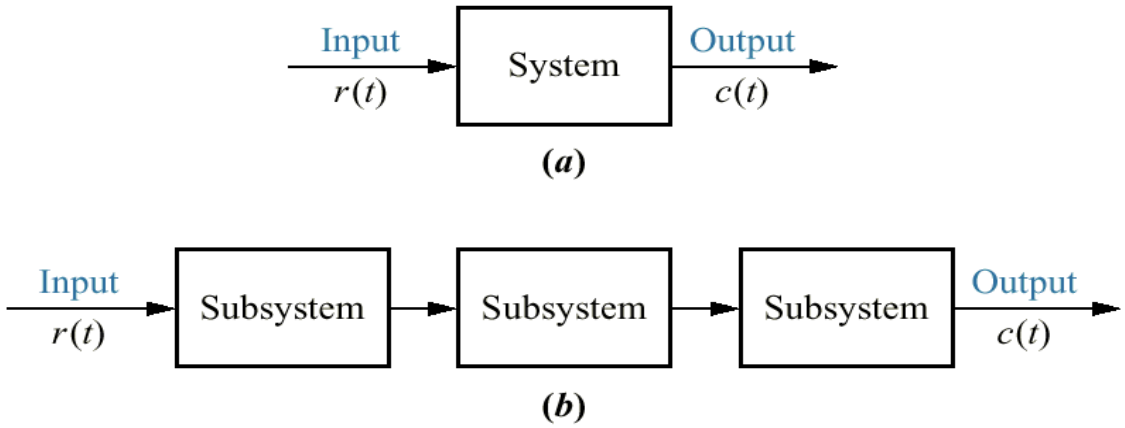
- One of the most important tasks in the analysis and design of control systems is mathematical modeling of the systems.
- The two most common methods of modeling linear systems are the transfer function method and the state-variable method.
- The transfer function is valid only for linear time-invariant systems
- The state equations can be applied to linear as well as nonlinear systems.
- A control system may be composed of various components including mechanical, thermal, fluid, pneumatic, and electrical; sensors and actuators; and computers.
- In this chapter, we review basic properties of these systems, otherwise known as **dynamic systems**.
- Using the basic modeling principles such as Newton's second law of motion o Kirchoff's law, the models of these dynamic systems are represented by differential equations.

2.1 Introduction

Systems schematic (assumptions) $\xrightarrow[\text{Laws}]{\text{Physical}}$ Mathematical model

- Transfer functions in frequency domain (Chapter 2)
- State equations in time domain (Chapter 3)

Physical laws: Newton's law, Ohm's law, Kirchoff's law,...



Note: The input, $r(t)$, stands for *reference input*.
The output, $c(t)$, stands for *controlled variable*.

2-4 LAPLACE TRANSFORM

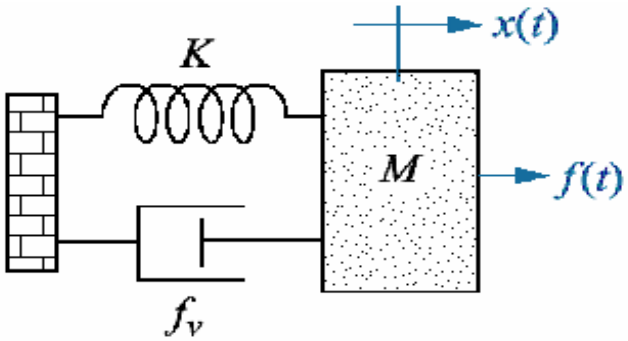
The Laplace transform is used to solve linear ordinary differential equations. It has the following two features:

1.The homogeneous equation and the particular integral of the solution of the differential equation are obtained in one operation.

1.The Laplace transform converts the differential equation into an algebraic equation in *s*-domain. It is then possible to manipulate the algebraic equation by simple algebraic rules to obtain the solution in the *s*-domain. The final solution is obtained by taking the inverse Laplace transform.

2.2 laplace Transform Review

Pre-example



$$f(t) = Inertia\ force + Damping\ force + Elastic\ force$$
$$f(t) = m\ddot{x} + f_v\dot{x} + kx$$

Differential equation of second order
Solution $x(t)$ depends on 2 initial conditions $x(0)$, $\dot{x}(t)$ and on the external input .

$$x(t) = \underbrace{Natural\ response}_{\text{Initial conditions}} + \underbrace{Forced\ response}_{\text{External input}}$$

Control: find suitable $f(t)$ to obtain desired $x(t)$, thus $f(t)$ should be a function of some measurements.

2.2 Laplace Transform Review

Basic Idea of Laplace Transform

Differential equation (time domain)

↓ Laplace Transform

Algebraic equation (frequency domain)
Analysis, manipulation, and design (frequency domain)

↓ Inverse Laplace Transform

Solution of the differential equation (time domain)

Laplace transform exists only for

1. Linear systems.
2. Time invarying systems (constant parameters)

2.2 Laplace Transform Review

1. $\frac{dy}{dx} + x^2 y = x$ linear.

2. $\frac{1}{x}\frac{d^2y}{dx^2} - y^3 = 3x$ not linear.

3. $\frac{dy}{dx} - \ln y = 0$ not linear.

4. $\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} + \frac{dy}{dx} = 2\sin x$ linear.

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1. $(\frac{d^3y}{dx^3})^4 + 2\frac{dy}{dx} = \sin x$ non linear.

2. $\frac{dy}{dx} - 2xy = x^2 - x$ linear.

3. $\frac{dy}{dx} - \sin y = -x$ non linear.

4. $\frac{d^2y}{dx} = 2xy$ linear.

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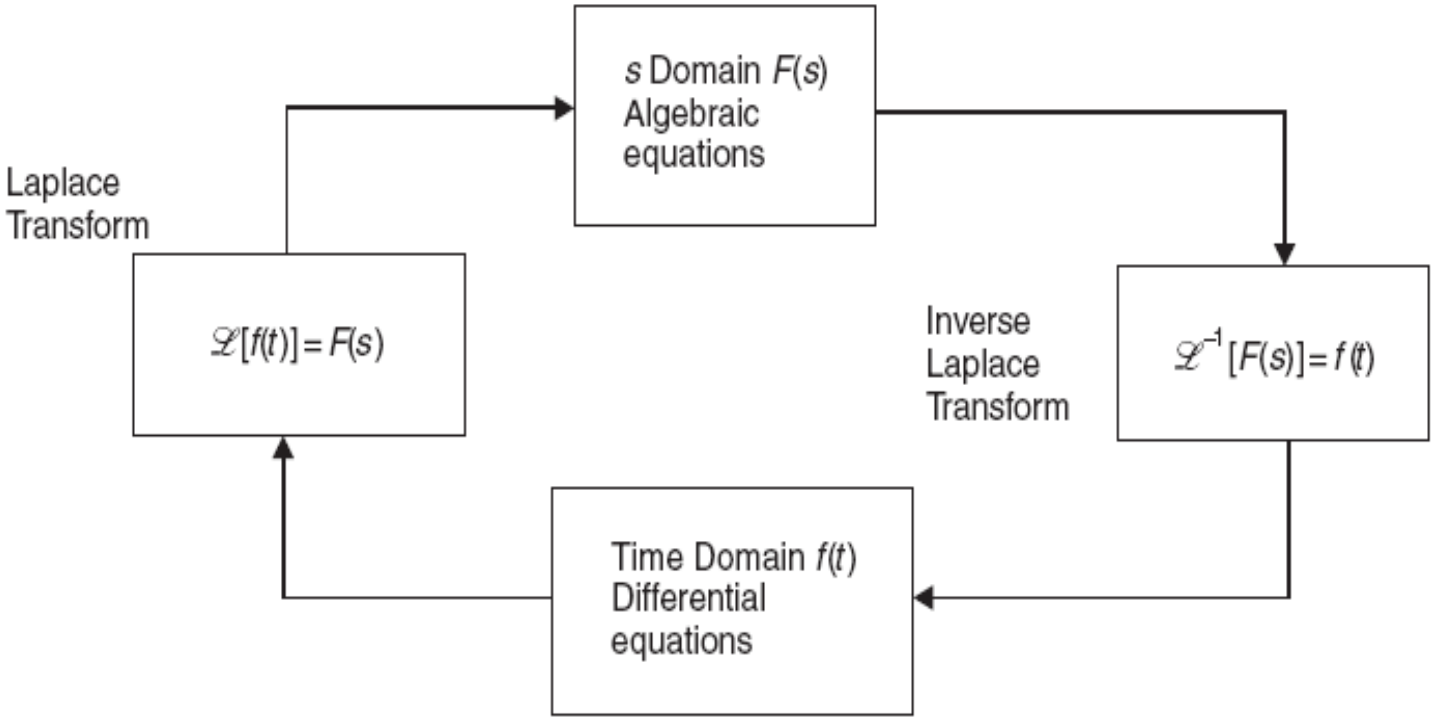
2.2 Laplace Transform Review



2.2 laplace Transform Review

• Using Laplace transforms we can represent the input, output, and system as separate entities.

• Their interrelationship will be algebraic.

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^{\infty} f(t)e^{-st}\,dt$$

where $s = \sigma + j\,\omega$

$$\mathcal{L}^{-1}\{F(s)\} = f(t) = \frac{1}{2\pi \mathrm{j}} \int_{c-\mathrm{j}\infty}^{c+\mathrm{j}\infty} F(s)e^{st}\,\mathrm{d}s$$

1. F is a complex-valued function of complex numbers
2. s is called the (complex) frequency variable, with units 1/sec; t is called the time variable (in sec); st is unitless

2.2 Laplace Transform Review

Item no.	f(t)	F(s)
1.	$\delta(t)$	1
2.	$u(t)$	$\frac{1}{s}$
3.	$tu(t)$	$\frac{1}{s^2}$
4.	$t^n u(t)$	$\frac{n!}{s^{n+1}}$
5.	$e^{-at}u(t)$	$\frac{1}{s+a}$
6.	$\sin \omega t u(t)$	$\frac{\omega}{s^2+\omega^2}$
7.	$\cos \omega t u(t)$	$\frac{s}{s^2+\omega^2}$

Table 2.1: Laplace transform table

2.2 Laplace Transform Review /example

- Find the Laplace Transform of $f(t) = be^{-at}\,u(t)$

$$F(s) = \int_0^{\infty} be^{-at}e^{-st}\,dt$$

$$= b\int_0^{\infty} e^{-(s+a)t}\,dt = -\frac{b}{s+a}e^{-(s+a)t}\Big|_0^{\infty}$$

$$= \frac{b}{s+a}$$

2.2 Laplace Transform Review

Item no.	Theorem	Name
1.	$\mathcal{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st}dt$	Definition
2.	$\mathcal{L}[kf(t)] = kF(s)$	Linearity theorem
3.	$\mathcal{L}[f_1(t) + f_2(t)] = F_1(s) + F_2(s)$	Linearity theorem
4.	$\mathcal{L}[e^{-at}f(t)] = F(s + a)$	Frequency shift theorem
5.	$\mathcal{L}[f(t - T)] = e^{-sT}F(s)$	Time shift theorem
6.	$\mathcal{L}[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right)$	Scaling theorem
7.	$\mathcal{L}\left[\frac{df}{dt}\right] = sF(s) - f(0-)$	Differentiation theorem
8.	$\mathcal{L}\left[\frac{d^2f}{dt^2}\right] = s^2F(s) - sf(0-) - \dot{f}(0-)$	Differentiation theorem
9.	$\mathcal{L}\left[\frac{d^nf}{dt^n}\right] = s^nF(s) - \sum_{k=1}^{n} s^{n-k}f^{k-1}(0-)$	Differentiation theorem
10.	$\mathcal{L}\left[\int_{0-}^{t} f(\tau)\,d\tau\right] = \frac{F(s)}{s}$	Integration theorem
11.	$f(\infty) = \lim_{s\to 0} sF(s)$	Final value theorem[1]
12.	$f(0+) = \lim_{s\to\infty} sF(s)$	Initial value theorem[2]

[1] For this theorem to yield correct finite results, all roots of the denominator of $F(s)$ must have negative real parts and no more than one can be at the origin.
[2] For this theorem to be valid, $f(t)$ must be continuous or have a step discontinuity at $t = 0$ (i.e., no impulses or their derivatives at $t = 0$).

Table 2.2: Laplace transform theorems

2.2 Laplace Transform Review /example

- Find the inverse Laplace transform of

$$? \Leftrightarrow 1/(s+3)^2$$

$$tu(t) \Leftrightarrow 1/s^2 \quad \text{Transform Table}$$

$$e^{-at} f(t) \Leftrightarrow F(s+a) \quad \text{Theorem Table}$$

$$e^{-at} tu(t) \Leftrightarrow 1/(s+a)^2$$

$$e^{-3t} tu(t) \Leftrightarrow 1/(s+3)^2$$

2.2 Laplace Transform Partial Fraction Expansion

When the Laplace transform solution of a differential equation is a rational function in s, it can be written as

$$G(s) = \frac{Q(s)}{P(s)}$$

where $P(s)$ and $Q(s)$ are polynomials of s. It is assumed that *the order of* $P(s)$ *in* s *is greater than that of* $Q(s)$. The polynomial $P(s)$ may be written

$$P(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$$

where $a_0, a_1, \ldots, a_{n-1}$ are real coefficients. The methods of partial-fraction expansion will now be given for the cases of simple poles, multiple-order poles, and complex-conjugate poles of $G(s)$.

Partial Fraction Expansion

Case 1: Roots of the denominator of F(s) are Real and Distinct .

G(s) Has Simple Poles If all the poles of $G(s)$ are simple and real, Eq. (2-117) can be written as

$$G(s) = \frac{Q(s)}{P(s)} = \frac{Q(s)}{(s+s_1)(s+s_2)\cdots(s+s_n)} \tag{2-143}$$

where $s_1 \neq s_2 \neq \cdots \neq s_n$. Applying the partial-fraction expansion, Eq. (2-143) is written

$$G(s) = \frac{K_{s1}}{s+s_1} + \frac{K_{s2}}{s+s_2} + \cdots + \frac{K_{sn}}{s+s_n} \tag{2-144}$$

The coefficient $K_{si}(i = 1, 2, \ldots, n)$ is determined by multiplying both sides of Eq. (2-143) by the factor $(s + s_i)$ and then setting s equal to $-s_i$. To find the coefficient K_{s1}, for instance, we multiply both sides of Eq. (2-143) by $(s + s_1)$ and let $s = -s_1$. Thus,

$$K_{s1} = \left[(s+s_1)\frac{Q(s)}{P(s)}\right]\Bigg|_{s=-s_1} = \frac{Q(-s_1)}{(s_2-s_1)(s_3-s_1)\cdots(s_n-s_1)} \tag{2-145}$$

2.2 Laplace Transform Review Partial Fraction Expansion

Case 1: Roots of the denominator of F(s) are Real and Distinct .

EXAMPLE 2-5-1 Consider the function

$$G(s) = \frac{5s+3}{(s+1)(s+2)(s+3)} = \frac{5s+3}{s^3+6s^2+11s+6} \tag{2-146}$$

which is written in the partial-fraction expanded form:

$$G(s) = \frac{K_{-1}}{s+1} + \frac{K_{-2}}{s+2} + \frac{K_{-3}}{s+3} \tag{2-147}$$

The coefficients K_{-1}, K_{-2}, and K_{-3} are determined as follows:

$$K_{-1} = [(s+1)G(s)]\Big|_{s=-1} = \frac{5(-1)+3}{(2-1)(3-1)} = -1 \tag{2-148}$$

$$K_{-2} = [(s+2)G(s)]\Big|_{s=-2} = \frac{5(-2)+3}{(1-2)(3-2)} = 7 \tag{2-149}$$

$$K_{-3} = [(s+3)G(s)]\Big|_{s=-3} = \frac{5(-3)+3}{(1-3)(2-3)} = -6 \tag{2-150}$$

Thus, Eq. (2-146) becomes

$$G(s) = \frac{-1}{s+1} + \frac{7}{s+2} - \frac{6}{s+3} \tag{2-151}$$

Toolbox 2-5-1

For Example 2-5-1, Eq. (2-146) is a ratio of two polynomials.

```
>> b = [5 3] % numerator polynomial coefficients
>> a = [1 6 11 6] % denominator polynomial coefficients
```

You can calculate the partial fraction expansion as

```
>> [r, p, k] = residue(b,a)
r =
    -6.0000
     7.0000
    -1.0000
p =
    -3.0000
    -2.0000
    -1.0000
k =
     []
```

Now, convert the partial fraction expansion back to polynomial coefficients.

```
>> [b,a] = residue(r,p,k)
b =
    0.0000    5.0000    3.0000
a =
    1.0000    6.0000   11.0000    6.0000
```

Partial Fraction Expansion

Case 1: Roots of the denominator of F(s) are Real and Distinct

- Find the inverse Laplace transform.

$$F(s)=\frac{1}{(s+1)(s+2)(s+4)}$$

$$=\frac{K_1}{s+1}+\frac{K_2}{s+2}+\frac{K_3}{s+4} \longrightarrow F(s)=\frac{1/3}{(s+1)}+\frac{1/2}{(s+2)}+\frac{1/6}{(s+4)}$$

$$K_1=\frac{1}{(-1+2)(-1+4)}=\frac{1}{3} \qquad f(t)=\frac{1}{3}e^{-t}+\frac{1}{2}e^{-2t}+\frac{1}{6}e^{-4t}$$

$$K_2=\frac{1}{(-2+1)(-2+4)}=\frac{1}{2}$$

$$K_3=\frac{1}{(-4+1)(-4+2)}=\frac{1}{6}$$

2.2 Laplace Transform Review /Partial Fraction Expansion

Case 2: Roots of the denominator of F(s) are Real and repeated

$$F(s)=\frac{N(s)}{D(s)}$$

$$=\frac{N(s)}{(s+p_1)^r(s+p_2)\ldots+(s+p_n)}$$

$$=\frac{K_1}{(s+p_1)^r}+\frac{K_2}{(s+p_1)^{r-1}}+\ldots+\frac{K_r}{(s+p_1)^+}$$

$$\frac{K_{r+1}}{(s+p_2)}+\ldots+\frac{K_n}{s+p_n}$$

2.2 Laplace Transform Review /Partial Fraction Expansion

Case 2: Roots of the denominator of F(s) are Real and repeated

$$F(s)=\frac{N(s)}{(s+p_1)^r(s+p_2)\dots(s+p_n)}$$

$$=K_1+(s+p_1)K_2+(s+p_1)^2K_3+\dots+(s+p_1)^{r-1}K_r+$$

$$\frac{(s+p_1)^r K_{r+1}}{(s+p_2)}+\dots+\frac{(s+p_1)^r K_n}{(s+p_n)}$$

$$K_i=\left[\frac{1}{(i-1)!}\ \frac{d^{i-1}F(s)}{ds^{i-1}}\right]_{s\to p_1} \qquad \begin{array}{l} i = 1, 2, \dots, r \\ 0! = 1 \end{array}$$

Case 2: Roots of the denominator of F(s) are Real and repeated

$$F(s)=\frac{2}{(s+1)(s+2)^2}=\frac{K_1}{s+1}+\frac{K_2}{(s+2)^2}+\frac{K_3}{s+2}$$

K_1 can be found as before to be $K_1=2$

For K_2 : multiply $F(s)$ by $(s+2)^2$

$$\frac{2}{s+1}=K_1\frac{(s+2)^2}{s+1}+K_2+K_3(s+2) \quad *$$

Let $s=-2 \rightarrow \frac{2}{-2+1}=K_2 \rightarrow K_2=-2$

Differentiate $(*)$

$$\frac{-2}{(s+1)^2}=\frac{K_1(s+2)s}{(s+1)^2}+K_3$$

Let $s=-2 \rightarrow \frac{-2}{(-2+1)^2}=0+K_3 \rightarrow K_3=-2$

Case 3: Roots of the denominator of F(s) are Complex or Imaginary

$$F(s)=\frac{3}{s(s^2+2s+5)}=\frac{K_1}{s}+\frac{K_2 s+K_3}{(s^2+2s+5)}$$

$$K_1=3/5; K_2=-3/5; K_3=-6/5$$

$$\sin(z)=\frac{e^{jz}-e^{-jz}}{2j}$$

$$\cos(z)=\frac{e^{jz}+e^{-jz}}{2}$$

$$\sinh(z)=\frac{e^{z}-e^{-z}}{2}$$

$$\cosh(z)=\frac{e^{z}+e^{-z}}{2}$$

The last term can be the sum of
[using Tab. 2.1 items 2 and 4]

$$\mathcal{L}[Ae^{-at}\cos\omega t]=\frac{A(s+a)}{(s+a)^2+\omega^2} \quad \text{or} \quad \mathcal{L}[Be^{-at}\sin\omega t]=\frac{B\omega}{(s+a)^2+\omega^2}$$

with $$\mathcal{L}[Ae^{-at}\cos\omega t+Be^{-at}\sin\omega t]=\frac{A(s+a)+B\omega}{(s+a)^2+\omega^2}$$

$$convert\ F(s)=\frac{3/5}{s}-\frac{3(s+2)}{5(s^2+2s+5)} \quad to\ F(s)=\frac{3/5}{s}-\frac{3[(s+1)+2(1/2)]}{5[(s+1)^2+2^2]}$$

Using Tab. 2.1 $$convert\ f(t)=3/5-\frac{3}{5}e^{-t}(\cos 2t+\frac{1}{2}\sin 2t)$$

2.2 Laplace Transform Review / Laplace transform solution of a differential equation

$$\frac{d^2y}{dt^2} + 12\frac{dy}{dt} + 32y = 32u(t)$$

with all initial conditions set to zero and step input $u(t)$.

Apply Laplace transform:

$$s^2Y(s) + 12sY(s) + 32Y(s) = \frac{32}{s}$$

$$Y(s) = \frac{32}{s(s^2 + 12s + 32)} = \frac{32}{s(s+4)(s+8)}$$

$$Y(s) = \frac{32}{s(s+4)(s+8)} = \frac{K_1}{s} + \frac{K_2}{(s+4)} + \frac{K_3}{(s+8)}$$

$$K_1 = \left.\frac{32}{(s+4)(s+8)}\right|_{s\to 0} = 1$$

$$K_2 = \left.\frac{32}{s(s+8)}\right|_{s\to -4} = -2$$

$$K_3 = \left.\frac{32}{s(s+4)}\right|_{s\to -8} = 1$$

$$Y(s) = \frac{1}{s} - \frac{2}{(s+4)} + \frac{1}{(s+8)}$$

Apply Laplace inverse transform:

$$y(t) = (1 - 2e^{-4t} + e^{-8t})u(t)$$

2.3 Transfer Function

• The Transfer Function algebraically relates a system's output to its input.

• A physical system that can be represented by a linear, time-invariant differential equation can be modeled as a transfer function.

$$a_n \frac{d^n y(t)}{dt^n} + a_{n-1} \frac{d^{n-1} y(t)}{dt^{n-1}} + \ldots + a_1 \frac{dy(t)}{dt} + a_o y(t) =$$

$$b_m \frac{d^m u(t-L)}{dt^n} + b_{m-1} \frac{d^{m-1} u(t-L)}{dt^{m-1}} + \ldots b_1 \frac{du(t-L)}{dt} + b_o u(t-L)$$

• $y(t)$ is the output, $u(t)$ is the input.

2.3 Transfer Function (2)

• Take the Laplace transform term-by-term and assume that all initial conditions are zero.

$$Y(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + ... + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + ... + a_1 s + a_0} e^{-sL} U(s)$$

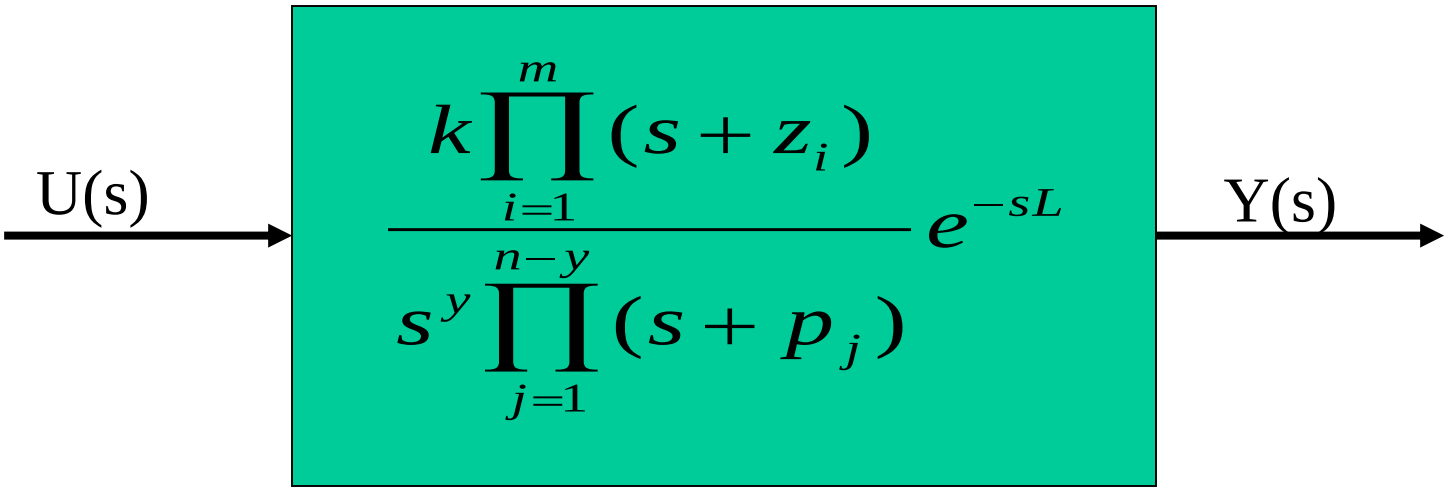
$$= \frac{k(s+z_1)(s+z_2)....(s+z_m)}{(s+p_1)(s+p_2)...(s+p_n)} e^{-sL} U(s)$$

Pole-zero transfer function form

$$= \frac{k \prod_{i=1}^{m} (s+z_i)}{s^y \prod_{j=1}^{n-y} (s+p_j)} e^{-sL} U(s)$$

Transfer function

2.3 Transfer Function (3)



Example 2.4

Transfer function for a differantial equation

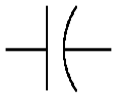
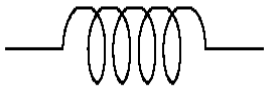
$$\frac{dc(t)}{dt} + 2c(t) = r(t)$$

Laplace transform:

$$sC(s) + 2C(s) = R(s)$$

$$G(s) = \frac{C(s)}{R(s)} = \frac{1}{s+2}$$

2.4 Electric Network Transfer Functions

Component	Voltage-Current	Current-Voltage	Voltage-Charge	Impedance	Admittance
 Capacitor	$v(t) = \frac{1}{C}\int_0^t i(\tau)\,d\tau$	$i(t) = C\frac{dv(t)}{dt}$	$v(t) = \frac{1}{C}q(t)$	$\frac{1}{Cs}$	Cs
Resistor	$v(t) = Ri(t)$	$i(t) = \frac{1}{R}v(t)$	$v(t) = R\frac{dq(t)}{dt}$	R	$\frac{1}{R} = G$
 Inductor	$v(t) = L\frac{di(t)}{dt}$	$i(t) = \frac{1}{L}\int_0^t v(\tau)d\tau$	$v(t) = L\frac{d^2q(t)}{dt^2}$	Ls	$\frac{1}{Ls}$

Note: The following set of symbols and units is used throughout this book: $v(t)$ = V (volts), $i(t)$ = A (amps), $q(t)$ = Q (coulombs), C = F (farads), R = Ω (ohms), G = ℧ (mhos), L = H (henries).

2.4 Electric Network Transfer Functions

- Passive Elements:
  - **Dissipative elements**: dissipate power
  - **Energy storage elements**: store energy
    - A-type energy storage elements: accumulate the across variable
    - T-type energy storage elements: accumulate the through variable

- Active elements:
  - **Sources**: generate power
    - A-type sources: sources of across variables
    - T-type sources: sources of through variables

Interact with other elements through one energy port and only require one pair of across and through variables

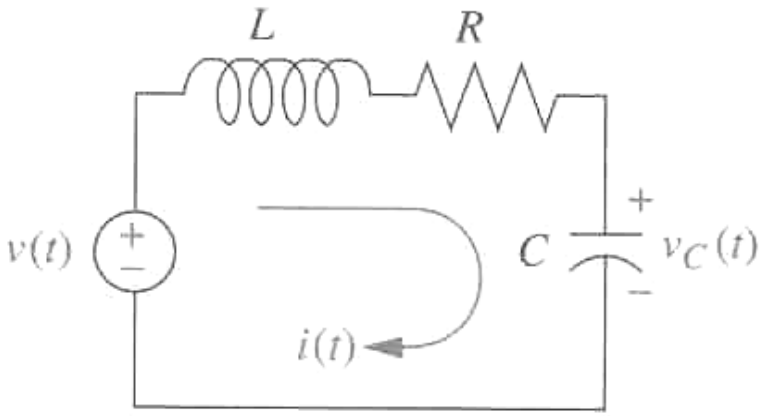
2.4 Electric Network Transfer Functions

- **Passive Elements**:
  - –Dissipative element:
    - D-type dissipative element: ***resistor***
  - –Storage elements:
    - A-type energy storage element: ***capacitor***
    - T-type energy storage element: ***inductor***

2.4 Electric Network Transfer Functions

Example:

$$\frac{V_c(s)}{V(s)} = ?$$



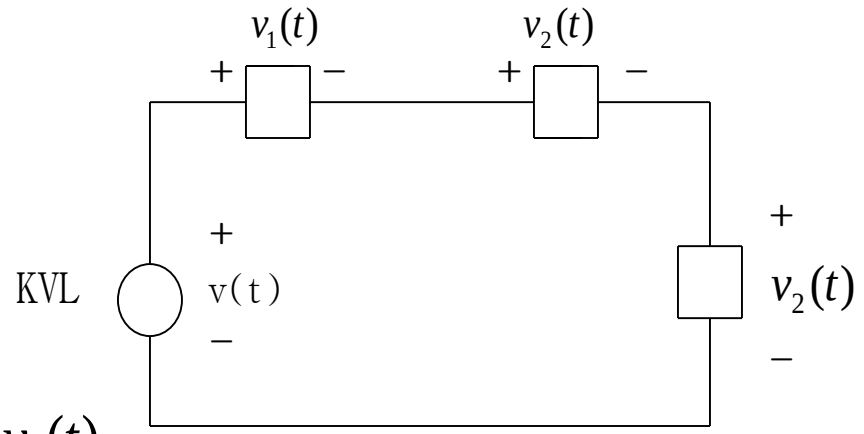
$$L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int_0^t i(\tau)d\tau = v(t)$$

$$i(t) = \frac{dq(t)}{dt}$$ Then: $$LC\frac{d^2v_c(t)}{dt^2} + RC\frac{dv_c(t)}{dt} + v_c(t) = v(t)$$

With Laplace-Tr.:

$$(LCs^2 + RCs + 1)V_c(s) = V(s) \Rightarrow \therefore \frac{V_c(s)}{V(s)} = \frac{1/LC}{s^2 + R/Ls + 1/LC}$$

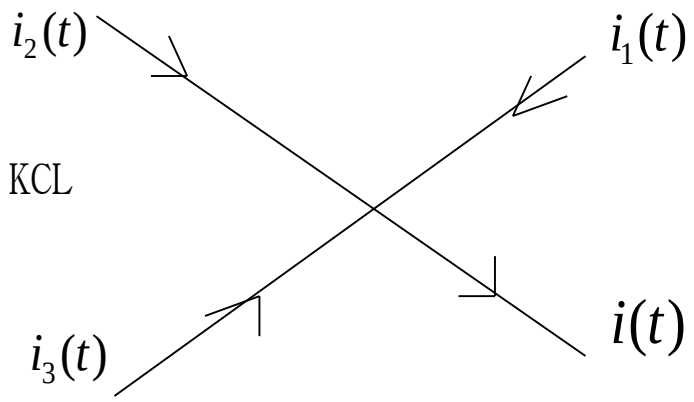
2.4 Electric Network Transfer Functions



$$v(t) = v_1(t) + v_2(t) + v_3(t)$$

$$\downarrow L$$

$$V(s) = V_1(s) + V_2(s) + V_3(s)$$



$$i(t) = i_1(t) + i_2(t) + i_3(t)$$

$$\downarrow L$$

$$I(s) = I_1(s) + I_2(s) + I_3(s)$$

2.4 Electric Network Transfer Functions

Ex 2.10 (p.57) Fig. 2.6 mesh analysis in s-domain

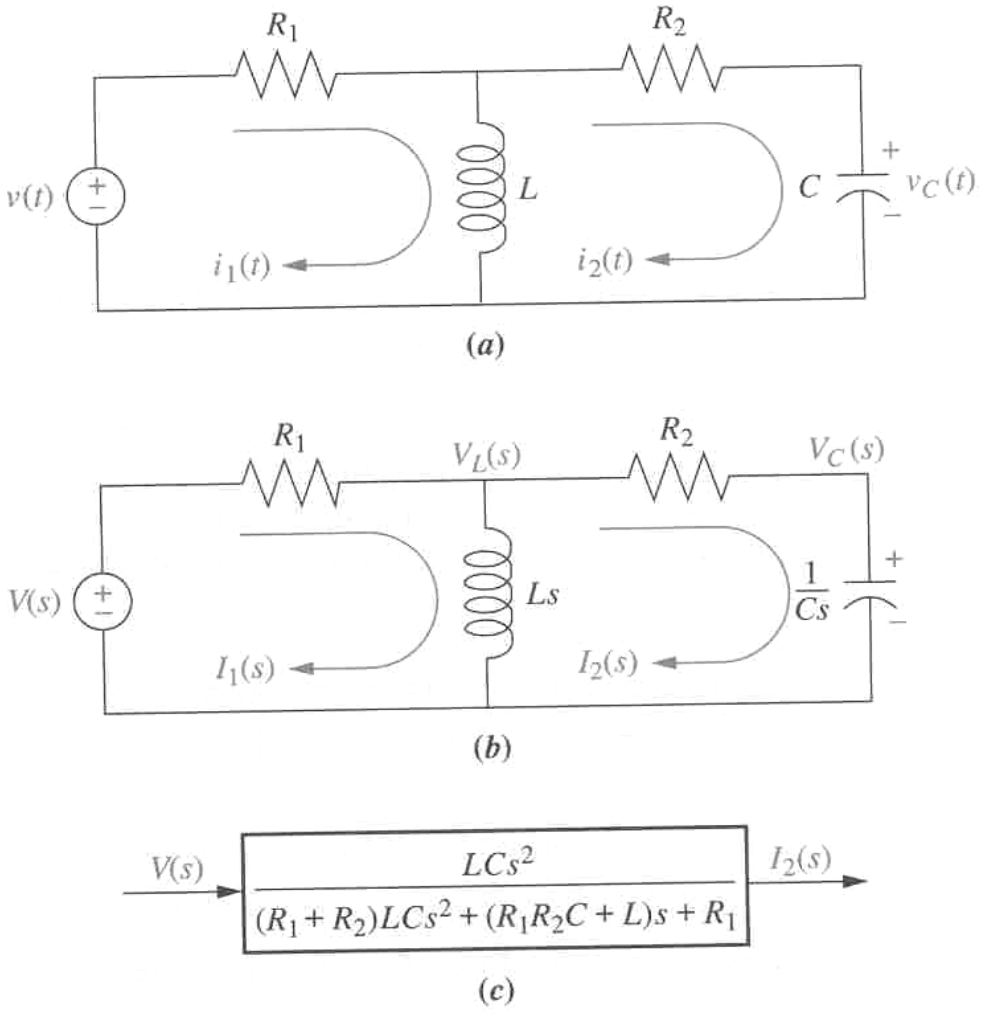


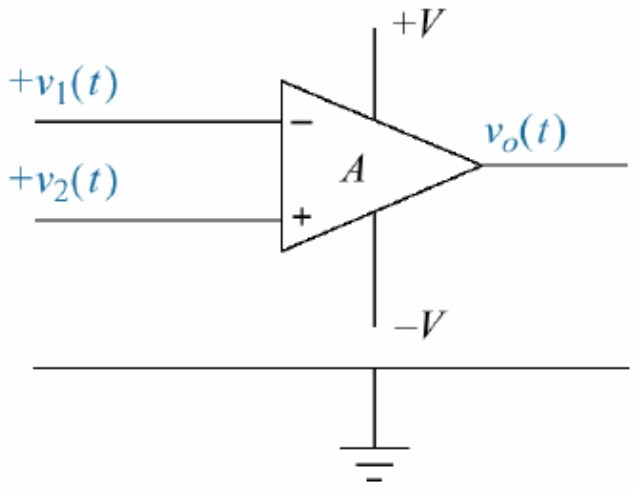
Figure 2.6

a.Two-loop electrical network; b.transformed Two-loop electrical network; c.block diagram

Operational Amplifiers:

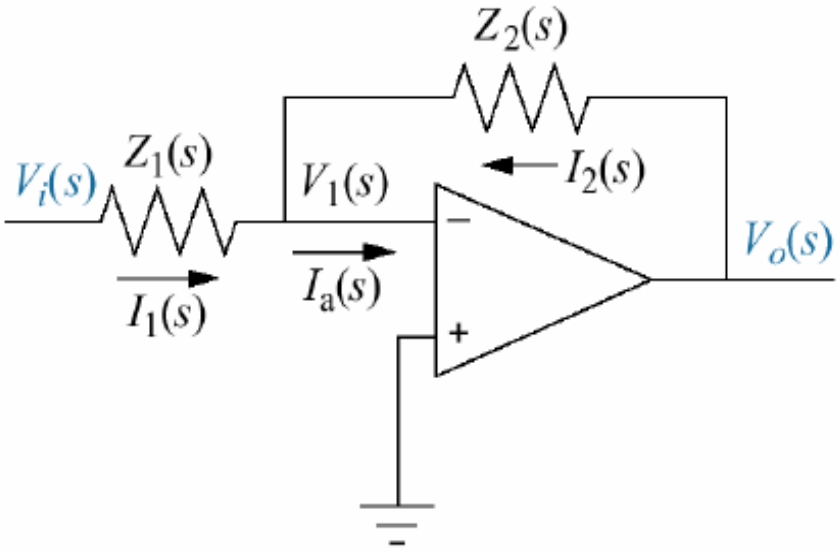
1. Differential input, $v_2(t) - v_1(t)$.
2. High input impedance, $Z_i = \infty$.
3. Low output impedance, $Z_o = 0$.
4. High constant gain amplification, $A = \infty$.

$$v_o(t) = A\left(v_2(t) - v_2(t)\right)$$

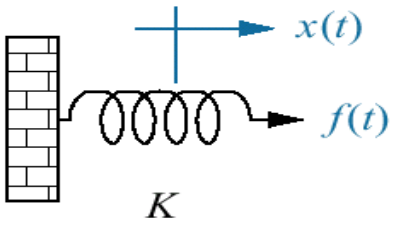
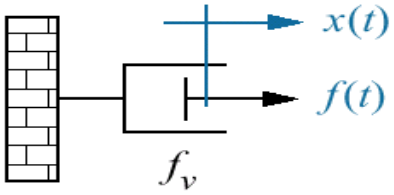
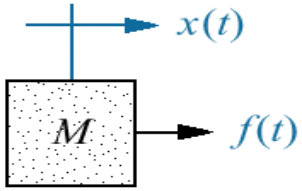


Inverting Operational Amplifiers

$$\frac{V_o(s)}{V_i(s)} = -\frac{Z_2(s)}{Z_1(s)}$$



2.5 Translational Mechanical System Transfer Functions

Component	Force-velocity	Force-displacement	Impedance $Z_M(s) = F(s)/X(s)$
Spring 	$f(t) = K\int_0^t v(\tau)\,d\tau$	$f(t) = Kx(t)$	K
Viscous damper 	$f(t) = f_v v(t)$	$f(t) = f_v \frac{dx(t)}{dt}$	$f_v s$
Mass 	$f(t) = M\frac{dv(t)}{dt}$	$f(t) = M\frac{d^2x(t)}{dt^2}$	Ms^2

Note: The following set of symbols and units is used throughout this book: $f(t)$ = N (newtons), $x(t)$ = m (meters), $v(t)$ = m/s (meters/second), K = N/m (newtons/meter), f_v = N-s/m (newton-seconds/meter), M = kg (kilograms = newton-seconds^2/meter).

Passive Elements:

– Dissipative element:

- D-type dissipative element: ***damper***

– Storage elements:

- A-type energy storage element: ***mass***
- T-type energy storage element: ***spring***

2.5 Translational Mechanical System Transfer Functions

• Translational mechanical systems can be modeled by a transfer function that algebraically relates the Laplace transform of the output to the Laplace transform of the input.

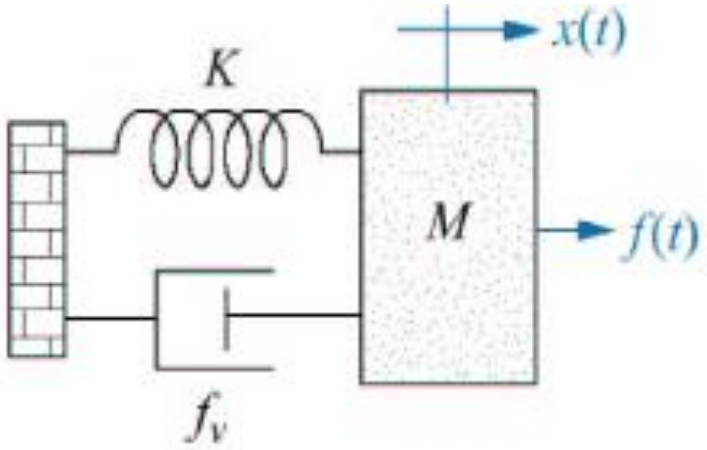
• Mechanical components
  – Spring constant, K
  – Coefficient of viscous friction, f_v
  – Mass, M

• We will write equations directly in terms of displacement. The input will be force, $f(t)$ and the output will be displacement, $x(t)$.

2.5 Translational Mechanical System Transfer Functions

Example:

Find the transfer function $X(s)/F(s)$.

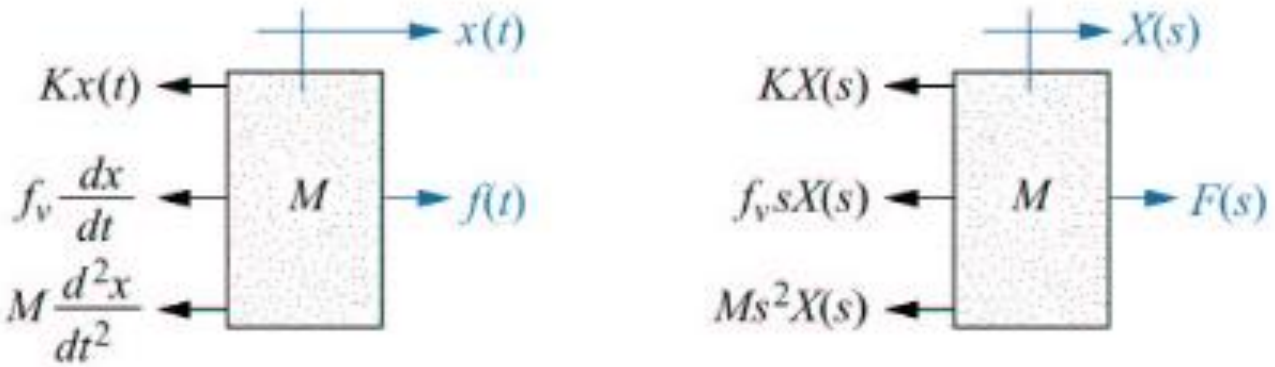


The first step is to draw the free-body diagram showing all of the forces felt by the mass. We assume that the mass is moving toward the right. Only the input force points to the right. All others point to the left. The sum of all forces is zero.

2.5 Translational Mechanical System Transfer Functions

Example:

Free-body diagram and its transform.



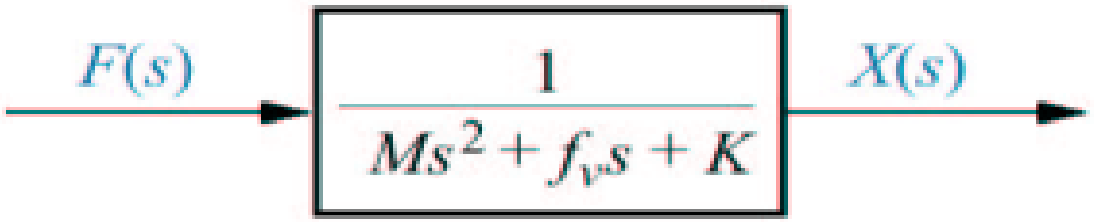
$$M\frac{d^2x(t)}{dt^2}+f_v\frac{dx(t)}{dt}+Kx(t)=f(t)$$

$$\left(Ms^2+f_vs+K\right)X(s)=F(s)$$

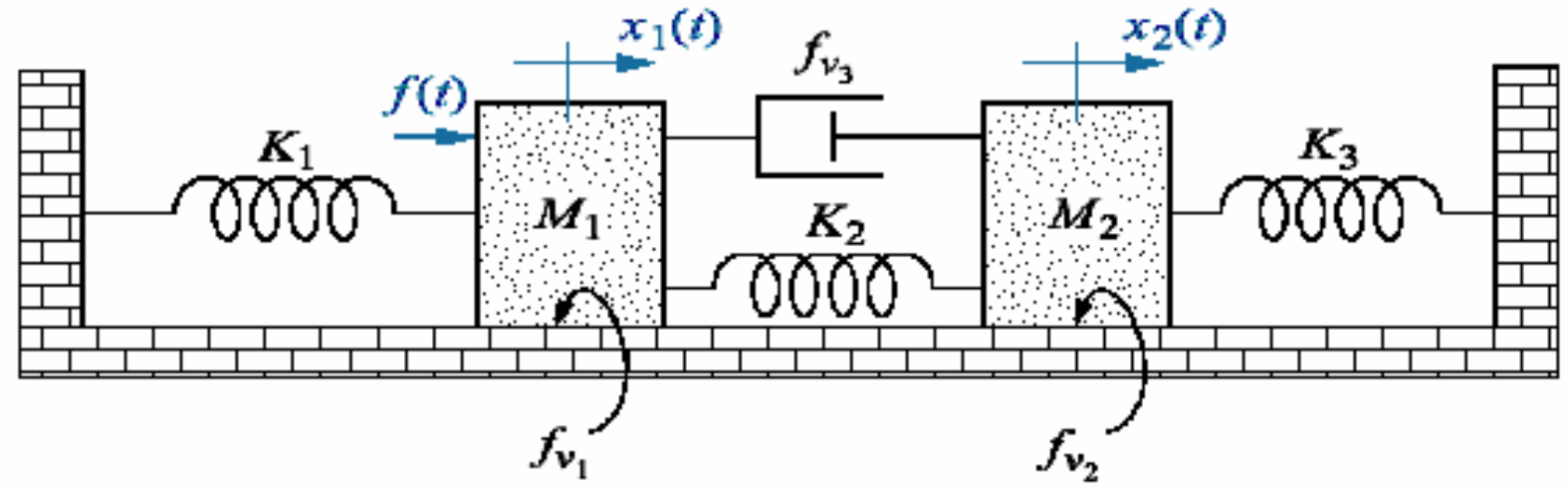
Example:

The resultant transfer function is

$$G(s)=\frac{X(s)}{F(s)}=\frac{1}{Ms^2+f_v s+K}$$



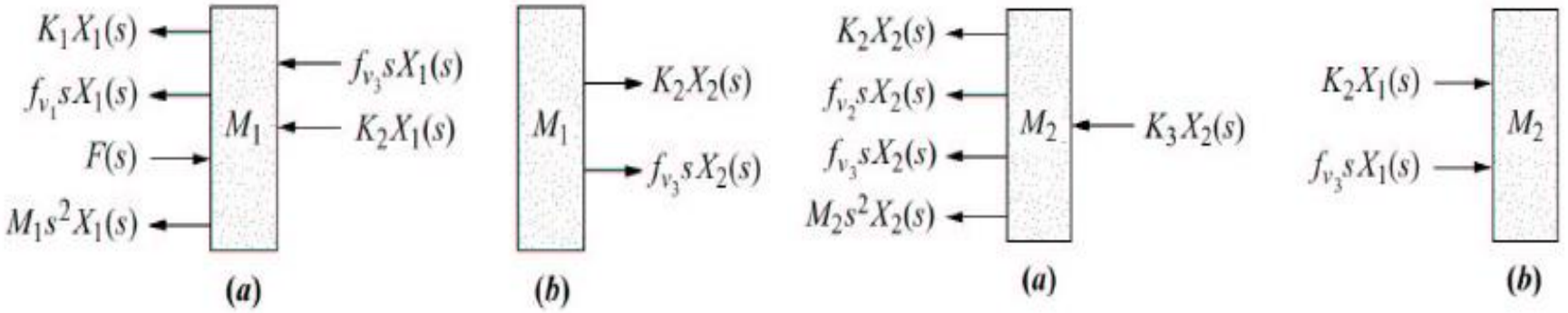
Find the transfer function, $X_2(s)/F(s)$.



Hold one mass still and use superposition to find the forces.

2.5 Translational Mechanical System Transfer Functions

Example:Find the transfer function, $X_2(s)/F(s)$.



Laplace Transform of the equations of motion.

$$\left[M_1s^2+(f_{v1}+f_{v2})s+K_1+K_2\right]X_1(s)-\left[f_{v3}s+K_2\right]X_2(s)=F(s)$$
$$-\left[f_{v3}s+K_2\right]X_1(s)+\left[M_2s^2+(f_{v2}+f_{v3})s+K_2+K_3\right]X_2(s)=0$$

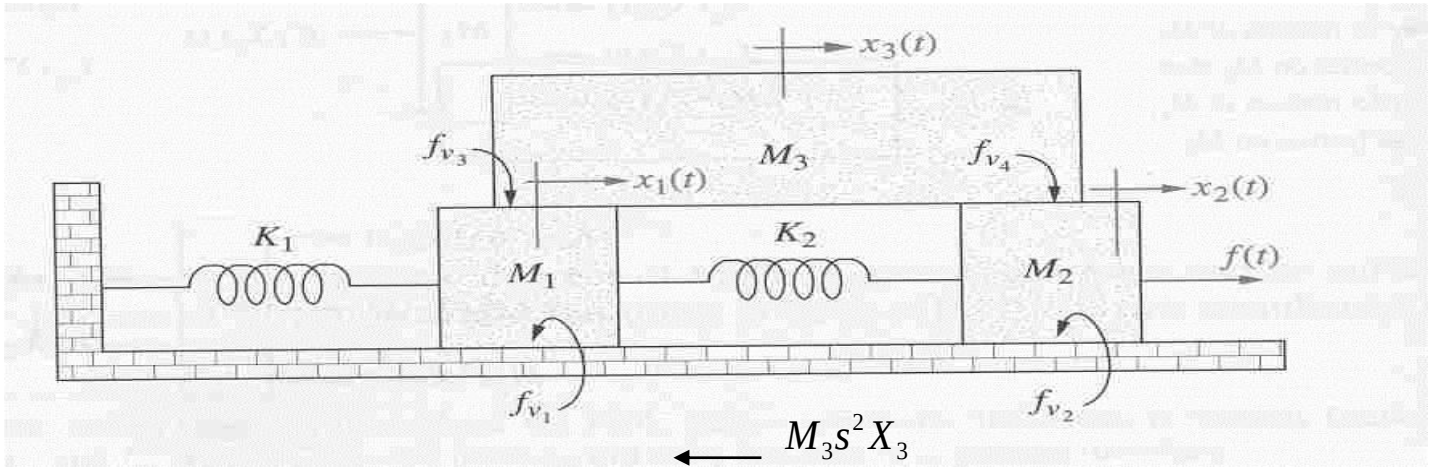
Cramer's rule produces the desired transfer function.

$$G(s)=\frac{X_2(s)}{F(s)}=\frac{(f_{v3}s+K_2)}{\Delta}$$

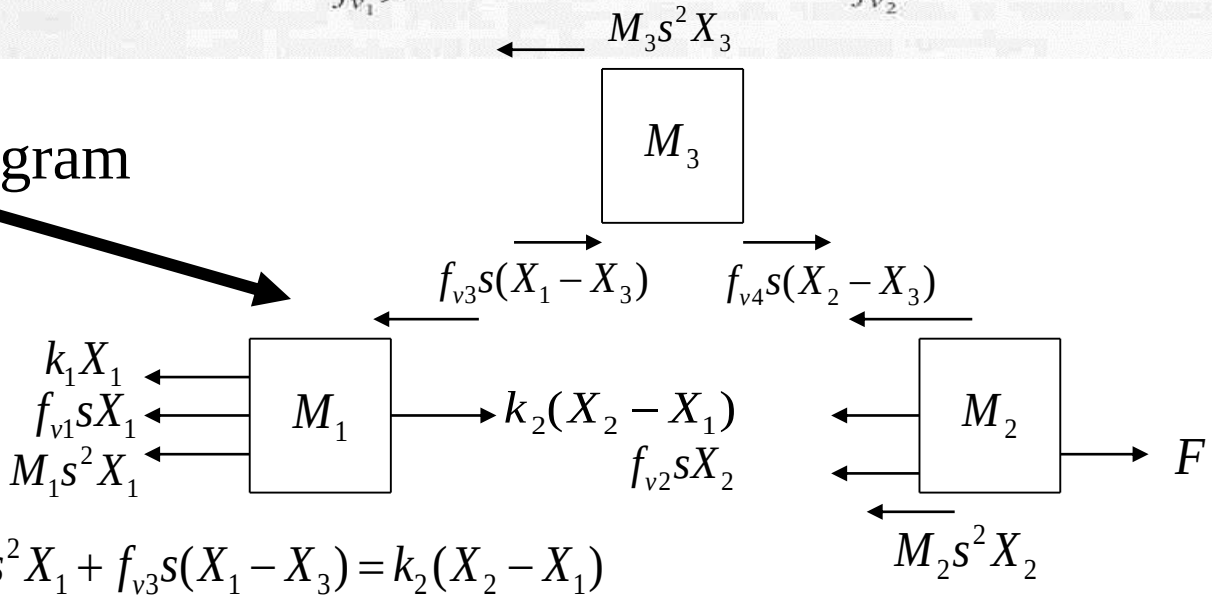
$$\Delta=\begin{vmatrix} M_1s^2+(f_{v1}+f_{v2})s+K_1+K_2 & -(f_{v3}s+K_2) \\ -(f_{v3}s+K_2) & M_2s^2+(f_{v2}+f_{v3})s+K_2+K_3 \end{vmatrix}$$

2.5 Translational Mechanical System Transfer Functions

Ex 2.18 (p.74)



Free Body Diagram

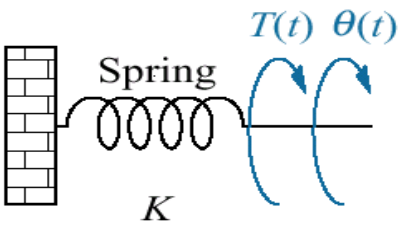
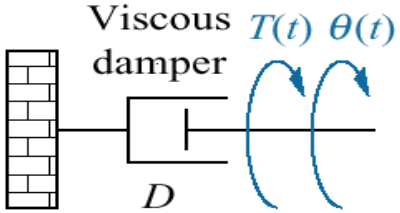
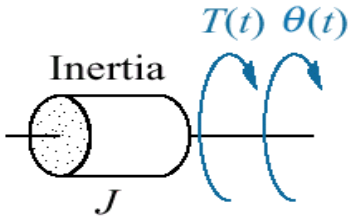


$$M_1 : \quad k_1X_1 + f_{v1}sX_1 + M_1s^2X_1 + f_{v3}s(X_1 - X_3) = k_2(X_2 - X_1)$$

$$M_2 : \quad k_2(X_2 - X_1) + f_{v2}sX_2 + M_2s^2X_2 + f_{v4}s(X_2 - X_3) = F$$

$$M_3 : \quad f_{v3}s(X_1 - X_3) + f_{v4}s(X_2 - X_3) = M_3s^2X_3$$

2.6 Rotational Mechanical System Transfer Functions

Component	Torque-angular velocity	Torque-angular displacement	Impedance $Z_M(s) = T(s)/\theta(s)$
	$T(t) = K\int_0^t \omega(\tau)\,d\tau$	$T(t) = K\theta(t)$	K
	$T(t) = D\omega(t)$	$T(t) = D\dfrac{d\theta(t)}{dt}$	Ds
	$T(t) = J\dfrac{d\omega(t)}{dt}$	$T(t) = J\dfrac{d^2\theta(t)}{dt^2}$	Js^2

Note: The following set of symbols and units is used throughout this book: $T(t)$ = N-m (newton-meters), $\theta(t)$ = rad (radians), $\omega(t)$ = rad/s (radians/second), K = N-m/rad (newton-meters/radian), D = N-m-s/rad (newton-meters-seconds/radian), J = kg-m^2 (kilogram-meters^2 = newton-meters-seconds^2/radian).

2.6 Rotational Mechanical System Transfer Functions

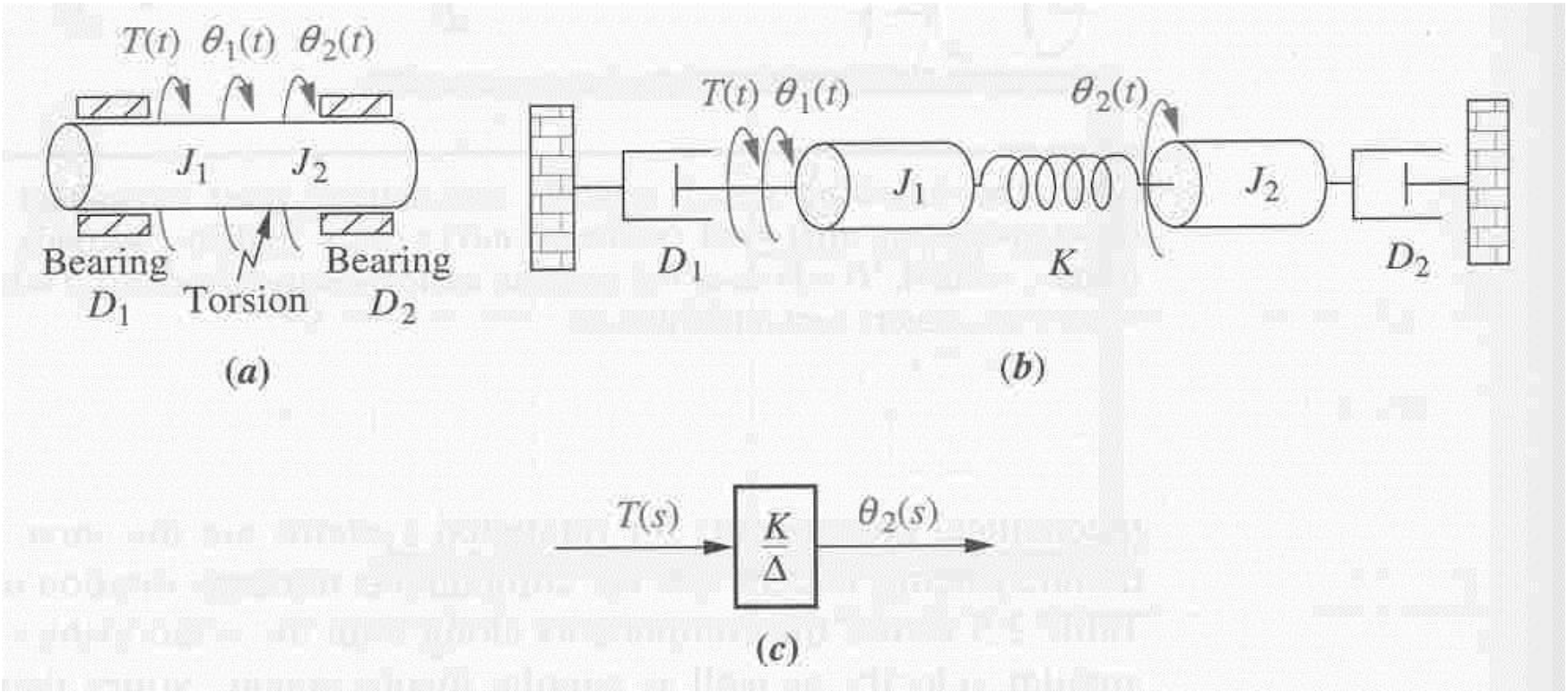


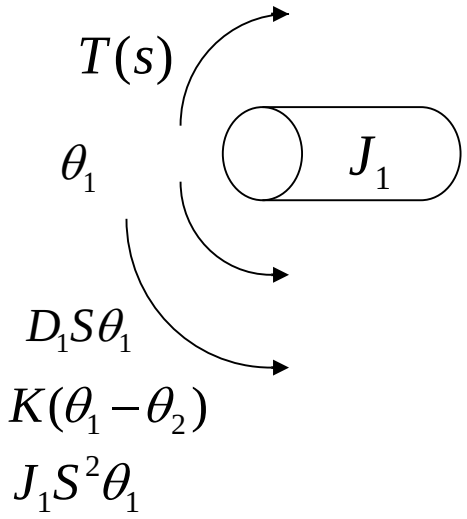
Figure 2.22

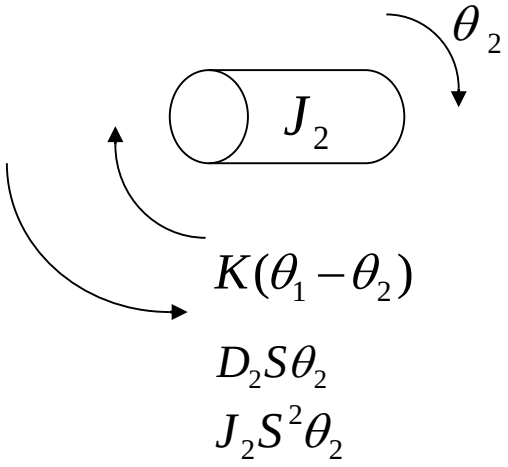
a. Physical system;

b. schematic;

c. block diagram

2.6 Rotational Mechanical System Transfer Functions





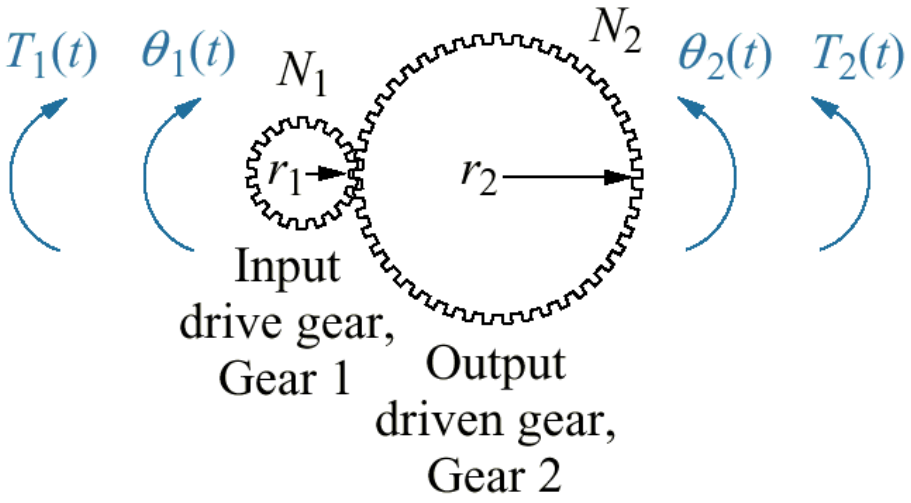
$$T = J_1S^2\theta_1 + K(\theta_1-\theta_2) + D_1S\theta_1$$

$$K(\theta_1-\theta_2) = D_2S\theta_2 + J_2S^2\theta_2$$

$$\therefore \frac{\theta_2(s)}{T(s)} = \quad (2.128)$$

2.7 Transfer Function for Systems with Gears

Gear Perform rotational and have the effect



Kinematics

$$v = r_1\dot{\theta}_1 = r_2\dot{\theta}_2$$

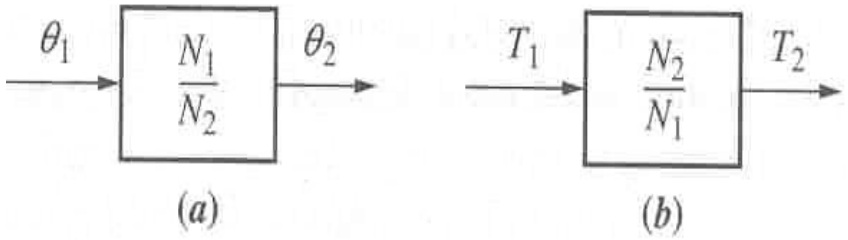
$$\frac{\dot{\theta}_2}{\dot{\theta}_1} = \frac{r_1}{r_2} = \frac{N_1}{N_2}$$

Torque

$$P = T_1\dot{\theta}_1 = T_2\dot{\theta}_2$$

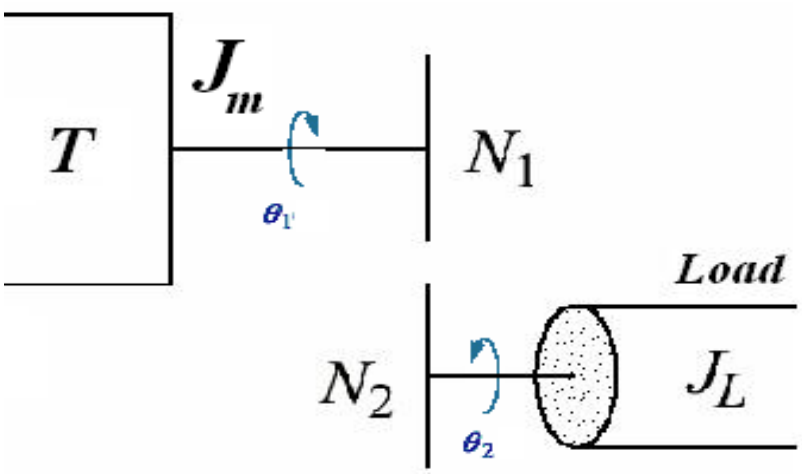
$$\frac{T_2}{T_1} = \frac{\dot{\theta}_1}{\dot{\theta}_2} = \frac{N_2}{N_1}$$

$\rightarrow P = Fv, \quad W = Fx, \quad P = W\dot{} \leftarrow$



2.7 Transfer Function for Systems with Gears

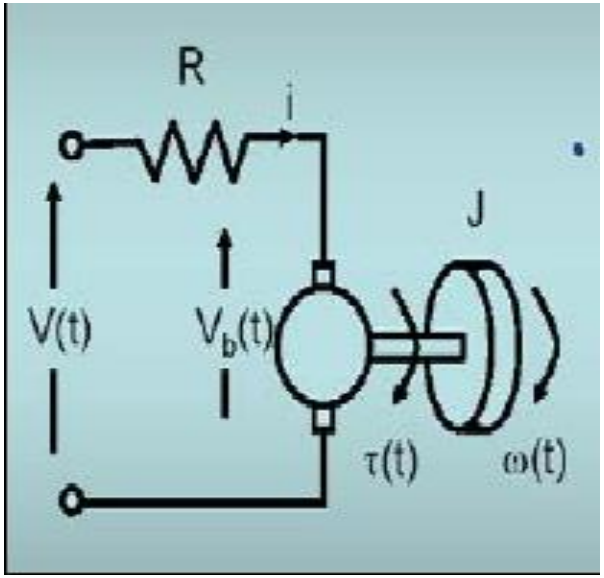
$$
\begin{aligned}
T &= \text{motor inertia} + \text{load inertia} \\
&= T_m + T_2 \text{ seen at } 1 \\
&= J_m \ddot{\theta}_1 + \left(J_L \ddot{\theta}_2\right)\frac{N_1}{N_2} \\
&= J_m \ddot{\theta}_1 + \left(J_L \ddot{\theta}_1 \frac{N_1}{N_2}\right)\frac{N_1}{N_2} \\
&= J_m \ddot{\theta}_1 + J_L \left(\frac{N_1}{N_2}\right)^2 \ddot{\theta}_1 \\
&= \left(J_m + \left(\frac{N_1}{N_2}\right)^2 J_L\right)\ddot{\theta}_1 \\
&= J_{Equivalent}\ddot{\theta}_1
\end{aligned}
$$



$$J_{Equivalent} = J_m + \left(\frac{N_1}{N_2}\right)^2 J_L$$

2.8 Electromechanical System Transfer Functions

A motor is an electromechanically component that yields a displacement output for a voltage input.

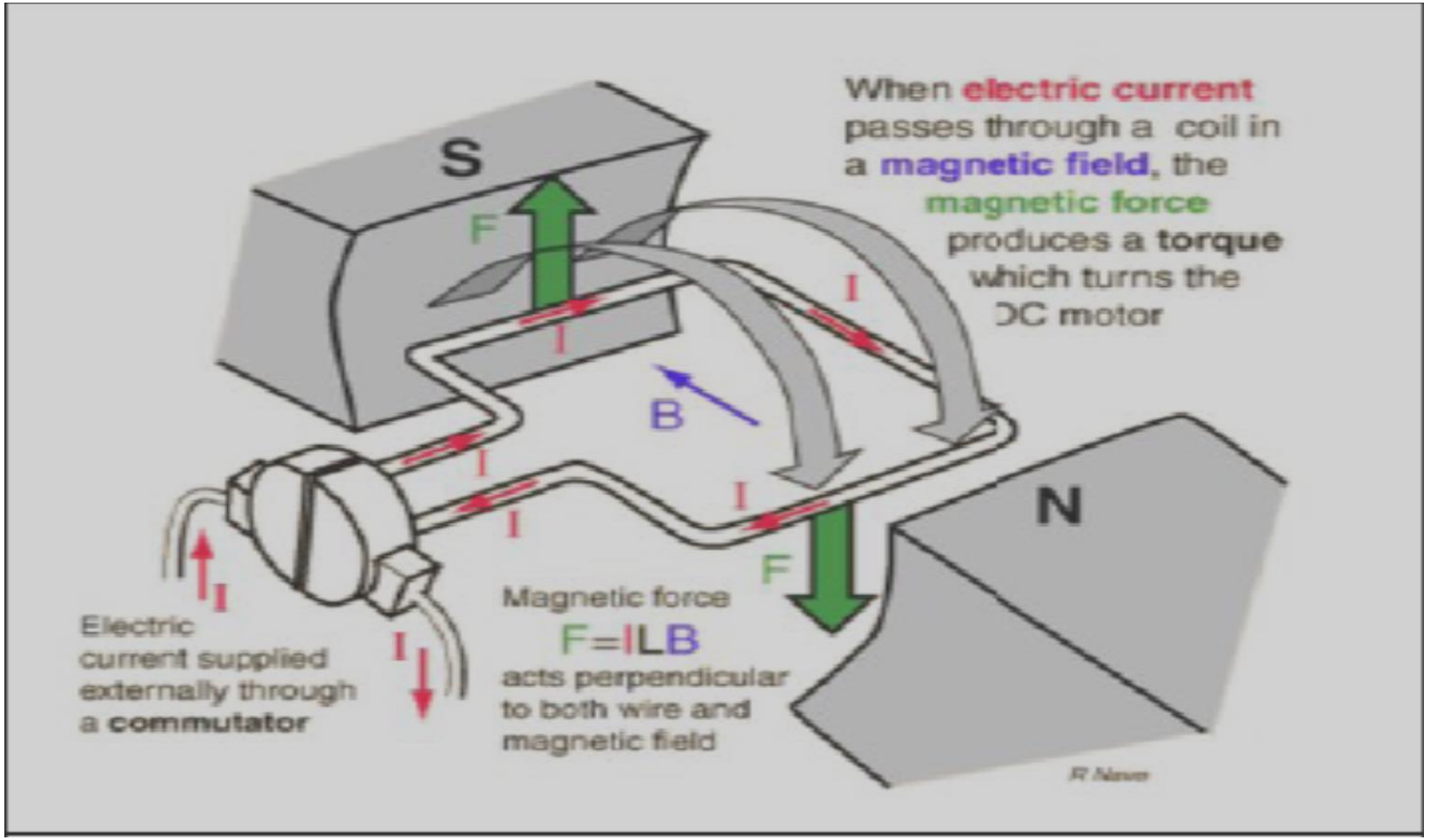


- Permanent magnet DC motor
- Interested in relationship between input voltage V(t) and angular position θ(t) or angular velocity $\omega(t) = \dot{\theta}$

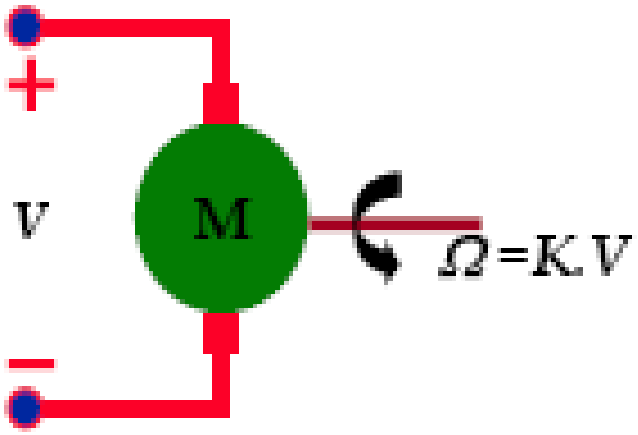
2.8 Electromechanical System Transfer Functions

Motor: A current passing through an inductor in a magnetic field generates force

Generator: A conductor moving at right angles to a magnetic field generates a voltage at the terminal of the conductor proportional to its speed


When electric current passes through a coil in a magnetic field, the magnetic force produces a torque which turns the DC motor
S
N
F
B
I
Electric current supplied externally through a commutator
Magnetic force
F=ILB
acts perpendicular to both wire and magnetic field

2.8 Electromechanical System Transfer Functions



DC-Motors

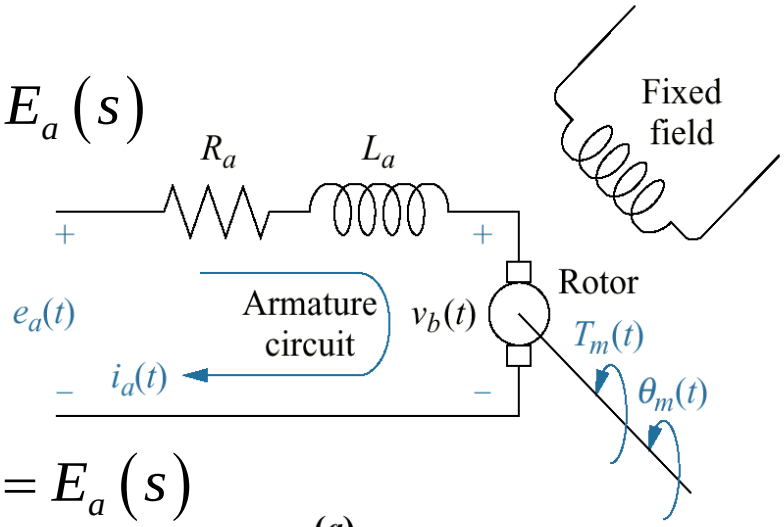
- Apply a voltage to armature
- Armature rotates in magnetic field
- Speed control by:
  - Armature voltage
  - Field Strength
- Speed proportional to Voltage
- Torque proportional to current
- Power=Speed x Torque

2.8 Electromechanical System Transfer Functions

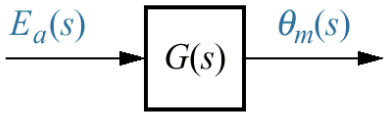
$$R_a I_a(s) + L_a s I_a(s) + V_b(s) = E_a(s)$$

$$T_m(s) = K_t I_a(s)$$

$$I_a(s) = \frac{1}{K_t} T_m(s)$$



(a)



(b)

$$\frac{(R_a + L_a s) T_m(s)}{K_t} + K_b s \theta_m(s) = E_a(s)$$

$$T_m(s) = (J_m s^2 + D_m s)\theta(s)$$

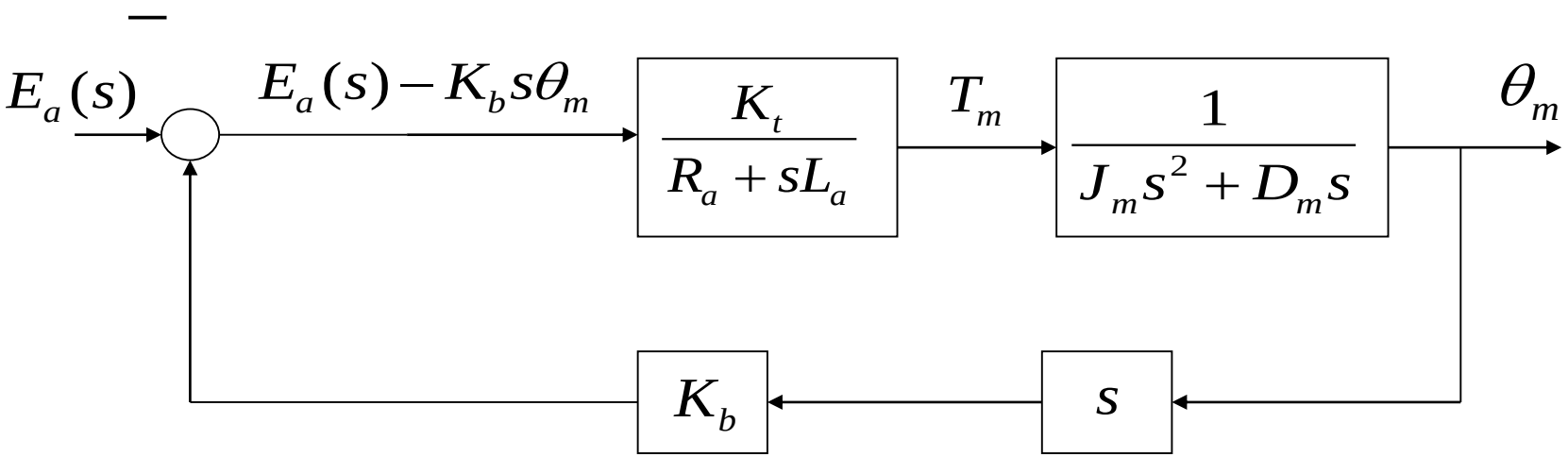
$$V_b(s) = K_b s \theta_m(s)$$

$$\frac{(R_a + L_a s)(J_m s^2 + D_m s)\theta_m(s)}{K_t} + K_b s \theta_m(s) = E_a(s)$$

Assuming that $L_a << R_a$

$$\left[\frac{R_a}{K_t}(J_m s + D_m) + K_b \right] s \theta_m(s) = E_a(s)$$

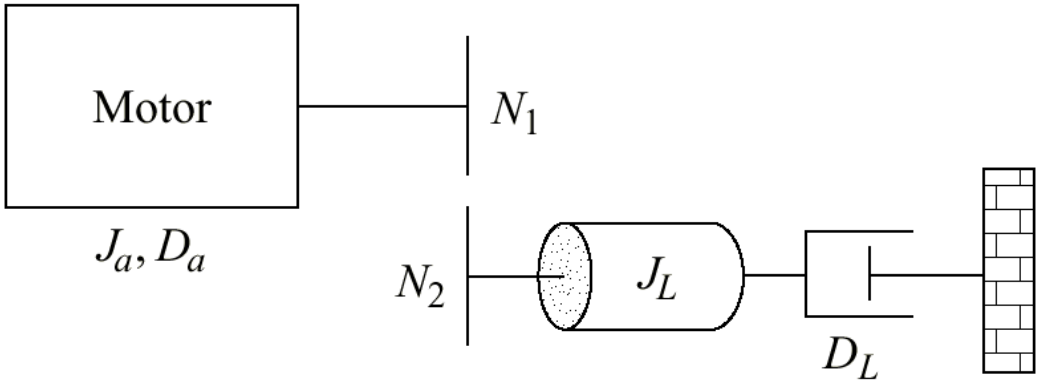
2.8 Electromechanical System Transfer Functions



$$\frac{\theta_m(s)}{E_a(s)} = \frac{K_t / R_a J_m}{s\left[s + \frac{1}{J_m}\left(D_m + \frac{K_t K_b}{R_a}\right)\right]}$$

$$\frac{\theta_m(s)}{E_a(s)} = \frac{K}{s(s+\alpha)}$$

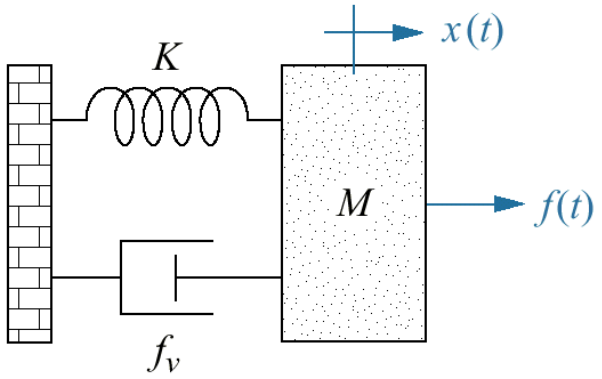
2.8 Electromechanical System Transfer Functions



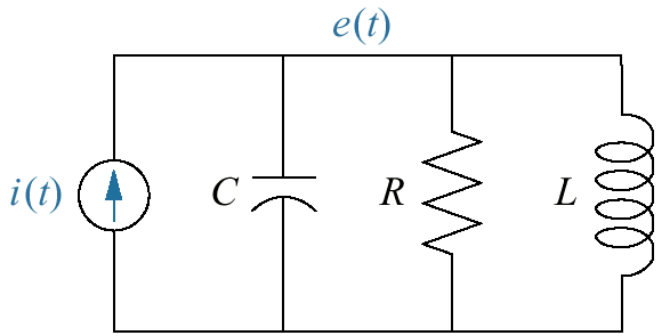
$$J_m = J_a + J_L\left(\frac{N_1}{N_2}\right)^2; \quad D_m = D_a + D_L\left(\frac{N_1}{N_2}\right)^2$$

2.9 Electrical Circuit Analogs

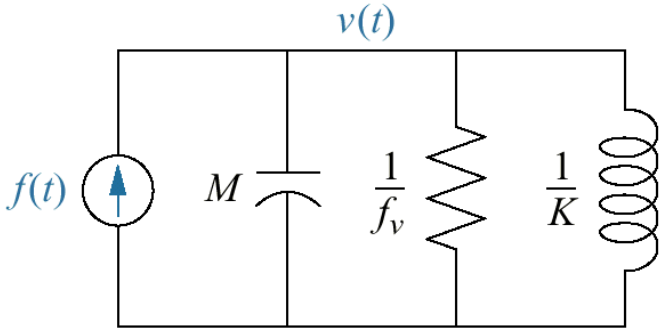
Series analog / Parallel analog



Mechanical system



Series analog



Parallel analog

Series analog

mass = M ⟶ capacitor = M farads

viscous damper = f_v ⟶ resistor = $\frac{1}{f_v}$ ohms

spring = K ⟶ inductor = $\frac{1}{K}$ henries

applied force = $f(t)$ ⟶ current source = $f(t)$

velocity = $v(t)$ ⟶ node voltage = $v(t)$

(*d*)

2.10 Nonlinearities

A linear system has two properties:

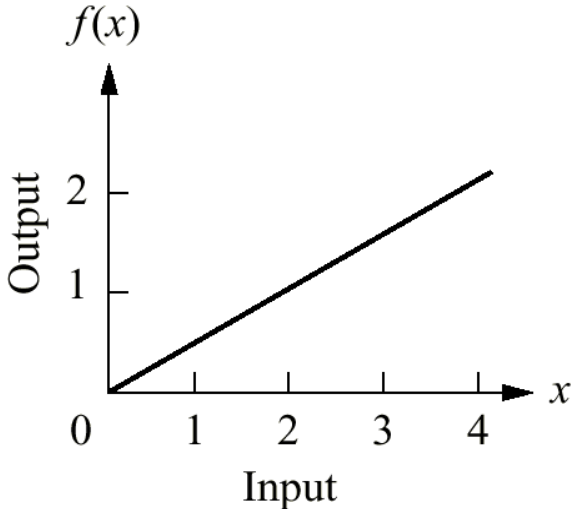
- Superposition:

$$\frac{\begin{array}{c} r_1(t) \to c_1(t) \\ r_2(t) \to c_2(t) \end{array}}{r_1(t) + r_2(t) \to c_1(t) + c_2(t)}$$

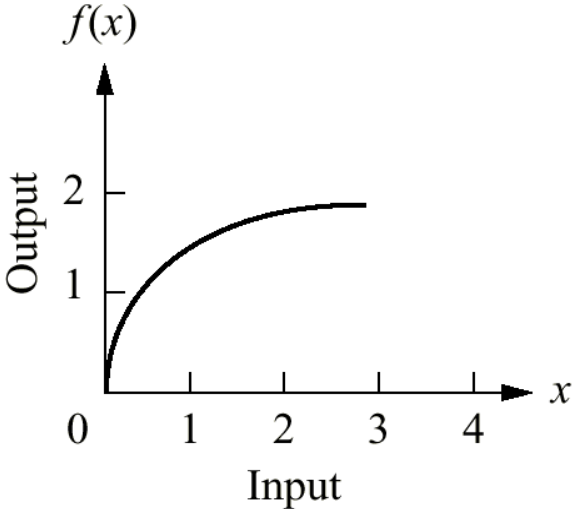
- Homogeniety:

$$\frac{r_1(t) \to c_1(t)}{A \cdot r_1(t) \to A \cdot c_1(t)}$$

Linear-/nonlinear System



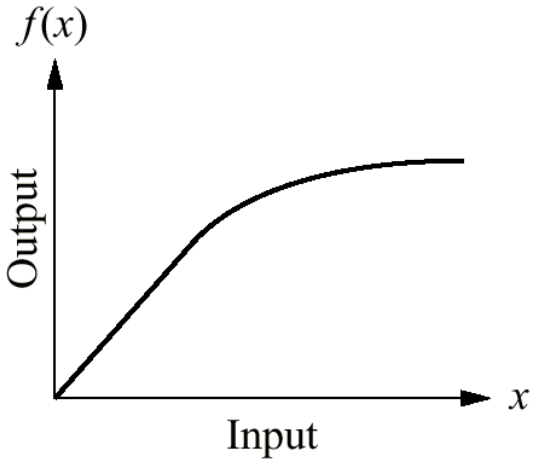
a. Linear system.



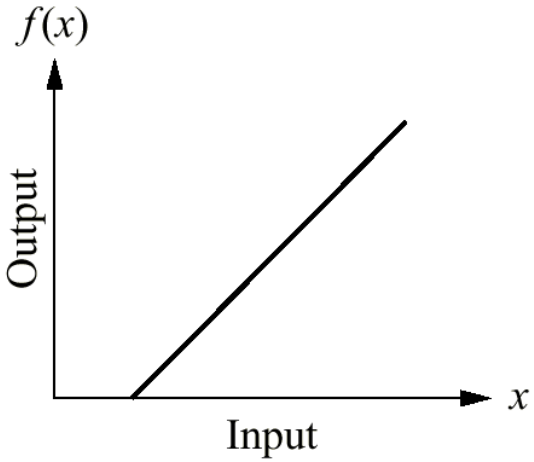
b. Nonlinear system

Some physical nonlinearity

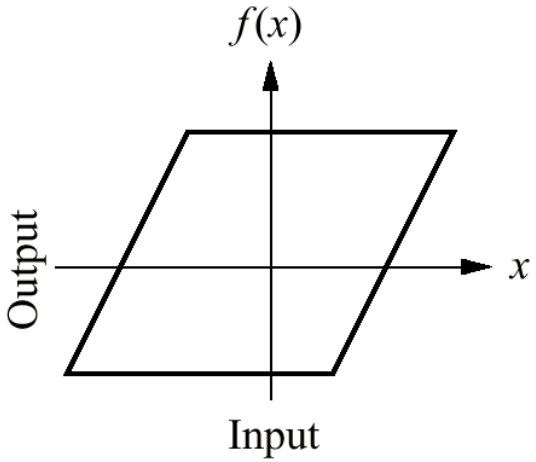
Amplifier saturation



Motor dead zone



Backlash in gears



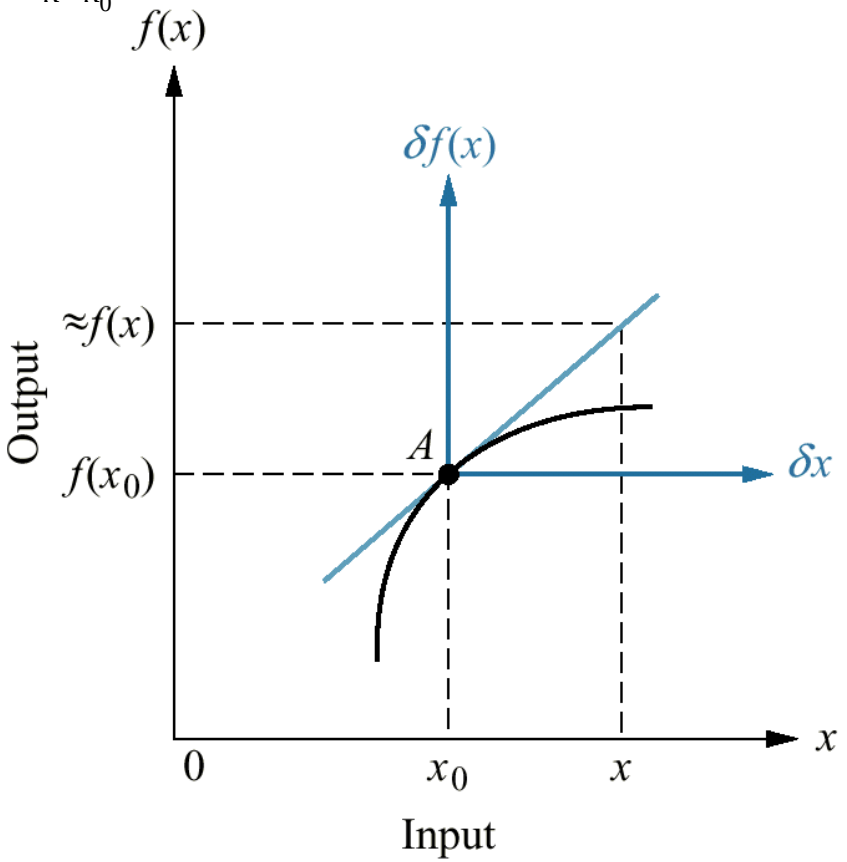
2.11 Linearization

A nonlinear function can be linearized by Taylor-series expansion (approximated by a linear function).

$$f(x) = f(x_0) + \left.\frac{df}{dx}\right|_{x=x_0} (x - x_0) + \left.\frac{d^2 f}{dx^2}\right|_{x=x_0} (x - x_0)^2 + \ldots$$

$$f(x) - f(x_0) \approx \left.\frac{df}{dx}\right|_{x=x_0} (x - x_0)$$

$$\delta f(x) \approx m|_{x=x_0}\, \delta x$$



Example : linearizing a differential equation

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + \cos(x) = 0$$

$\cos(x)$ is nonlinear, linarize it about $x = \frac{\pi}{4}$

$$\cos(x) \approx \cos\left(\frac{\pi}{4}\right) - \left(\sin(x_0)\right)(x - x_0)$$

$$\approx \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}(\bar{x})$$

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}(\bar{x}) = 0$$

but $(\bar{x}) = x - x_0$

$$\frac{d\bar{x}}{dt} = \frac{dx}{dt}, \text{ and } \frac{d^2\bar{x}}{dt^2} = \frac{d^2x}{dt^2}$$

$$\therefore \frac{d^2\bar{x}}{dt^2} + 2\frac{d\bar{x}}{dt} - \frac{\sqrt{2}}{2}\bar{x} = -\frac{\sqrt{2}}{2}$$

Partial Fraction Expansion

Case 1: Roots of the denominator of F(s) are Real and Distinct

- Find the inverse Laplace transform.

$$F(s)=\frac{N(s)}{D(s)}$$

$$=\frac{N(s)}{a_n s^n+a_{n-1} s^{(n-1)}+\ldots+a_0}$$

$$=\frac{N(s)}{(s+P_1)(s+P_2)\ldots(s+P_m)(s+P_n)}$$

$$=\frac{K_1}{s+p_1}+\frac{K_2}{s+p_2}+\ldots+\frac{K_m}{s+p_m}+\frac{K_n}{s+p_n}$$

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$$(s+p_m)F(s)=(s+p_m)\frac{K_1}{s+p_1}+(s+p_m)\frac{K_2}{s+p_2}+$$

$$\ldots+K_m+(s+p_m)\frac{K_n}{s+p_n}$$

$$K_m=\frac{N(p_m)}{(-p_m+p_1)(-p_m+p_2)\ldots(-p_m+p_n)}$$